Analyses of Avoidance Manipulability for Redundant Manipulators

*Tongxiao ZHANG (University of Fukui), Mamoru MINAMI (University of Fukui), Osami YASUKURA (University of Fukui), Wei SONG (University of Fukui), Yusaku NAKAMURA (University of Fukui)

Abstract— This paper is concerned with a new concept of avoidance manipulability inspired from manipulability, which represents the shape-changing ability of each intermediate link. Based on avoidance manipulability, we present the "Local Non-singular Configuration Assumptions". By analyses of avoidance manipulability for redundant manipulators, we think that it gives the foundation to assess the shape-changing ability to improve the structure at the first step of design for a new robot.

Key Words: Redundant manipulator, Avoidance manipulability, Partial non-singular configuration

1. Introduction

A variety of indices have been proposed for evaluation of the performance of robot manipulators since the mid-1980s. Up to now, the manipulability ellipsoid [1] was presented to evaluate the static performance of a robot manipulator as an index evaluating the manipulator's shape on the view point of how much the hand velocity can be generated by normalized joint velocity, that is, the velocity generation ability of the end-effector.

Many researches were an argument in a condition that an assumption guarantees the possibility that the avoiding motion could be realized. However, facing the situation that the moving obstacle appears suddenly near the manipulator, it requires the manipulator to possess the ability to avoid this moving obstacle suddenly appearing by changing its shape quickly, which is so-called "shape-changing ability". In this background, we firstly present the avoidance manipulability ellipsoid concept as an index evaluating shape-changing ability of the manipulator, which is inspired from the manipulability ellipsoid. Shape-changing ability is just related to peculiar characteristic of manipulator such as each link's length and whole manipulator's configuration, and it is independent of the shape of obstacles. Based on avoidance manipulability, we present the "Local Non-singular Configuration Assumptions".

2. Avoidance Manipulability

2·1 Jacobian Matrix

Representing the position vector of each link by $\boldsymbol{r}_{p,i} \in R^{m_p}$ and representing the orientation vector of each link by $\boldsymbol{r}_{o,i} \in R^{m_o}$. Here, m_p denotes the position dimension number of working space ($2 \le m_p \le 3$), m_o denotes the orientation dimension number of working space ($0 \le m_o \le 3$). $m = m_p + m_o$ and n denotes the number of the manipulator's links, $i = 1, 2, \cdots, n$, and $m < n$ because of redundancy. The vector and Jacobian matrix of i-th link including both position and orientation dimension space can be defined as

$$\boldsymbol{r}_i = \begin{bmatrix} \boldsymbol{r}_{p,i} \\ \boldsymbol{r}_{o,i} \end{bmatrix} \}m \tag{1}$$

and

$$\begin{aligned} \boldsymbol{J}_i &= \begin{bmatrix} \boldsymbol{J}_{p,i} \\ \boldsymbol{J}_{o,i} \end{bmatrix} \\ &= [\underbrace{\tilde{\boldsymbol{j}}_{i,1}, \ \cdots, \ \tilde{\boldsymbol{j}}_{i,i}}_{i}, \underbrace{\boldsymbol{0}}_{n-i}] \}m \\ &= [\tilde{\boldsymbol{J}}_i, \ \boldsymbol{0}] \end{aligned} \tag{2}$$

Here, the detailed formations of $\boldsymbol{r}_i$ and $\boldsymbol{J}_i$ are skipped.

2·2 Avoidance Matrix

Here, we define the first avoidance matrix ${}^1\boldsymbol{M}_i$ ($i = 1, 2, \cdots, n-1$) as

$$\begin{aligned} {}^1\boldsymbol{M}_i &= \boldsymbol{J}_i(\boldsymbol{I}_n - \boldsymbol{J}_n^+ \boldsymbol{J}_n) \\ &= \boldsymbol{J}_i \boldsymbol{L}_n \end{aligned} \tag{3}$$

If $rank(\boldsymbol{J}_n) = m$. $\boldsymbol{L}_n$ can be decomposed by

$$\begin{aligned} \boldsymbol{L}_n &= \boldsymbol{I}_n - \boldsymbol{J}_n^+ \boldsymbol{J}_n = \boldsymbol{I}_n - \boldsymbol{V}\boldsymbol{\Sigma}^+ \boldsymbol{U}^T \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^T \\ &= \boldsymbol{V}\boldsymbol{V}^T - \boldsymbol{V} \begin{matrix} \\ m \\ n-m \end{matrix} \overset{\begin{matrix} m & n-m \end{matrix}}{\begin{pmatrix} \boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}} \boldsymbol{V}^T \\ &= \boldsymbol{V} \begin{matrix} m \\ n-m \end{matrix} \overset{\begin{matrix} m & n-m \end{matrix}}{\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-m} \end{pmatrix}} \boldsymbol{V}^T \\ &= n \overset{\begin{matrix} m & n-m \end{matrix}}{\begin{pmatrix} \boldsymbol{V}_m & \boldsymbol{V}_{n-m} \end{pmatrix}} \begin{matrix} m \\ n-m \end{matrix} \overset{\begin{matrix} m & n-m \end{matrix}}{\begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{n-m} \end{pmatrix}} \boldsymbol{V}^T \\ &= n \overset{n-m}{\begin{pmatrix} \boldsymbol{V}_{n-m} \end{pmatrix}} \ n-m \overset{n}{\begin{pmatrix} \boldsymbol{V}_{n-m}^T \end{pmatrix}} \end{aligned} \tag{4}$$

In (4), $\boldsymbol{V}$ is $n \times n$ orthogonal matrix, $\boldsymbol{U}$ is $m \times m$ orthogonal matrix, $\boldsymbol{\Sigma}$ is $m \times n$ matrix, which includes a diagonal matrix composing of non-zero singular values of $\boldsymbol{J}_n$ and the rest parts are zero elements.

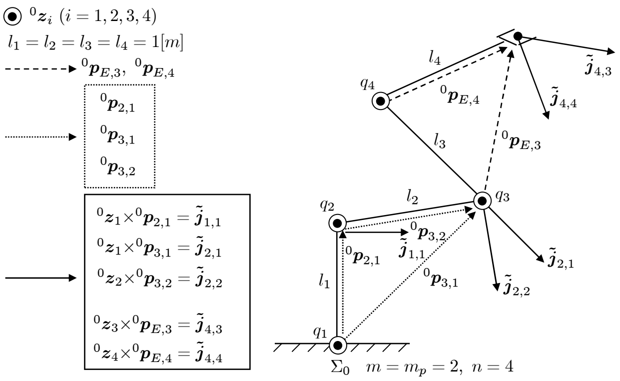

Fig.1 A simple example used to explain "*Assumptions*"

3. $rank(^1\boldsymbol{M}_i)$ $\quad (i=1,2,\cdots,n-1)$

3·1 Local Non-singular Configuration Assumptions

"Local Non-singular Configuration Assumptions"are as

$$\begin{cases} (a): \ rank(\tilde{\boldsymbol{J}}_n{}^{n-m+1\to n}) = m \\ (b): \ rank(\tilde{\boldsymbol{J}}_i) = min\{i,m\} \end{cases} \tag{5}$$

In (5), $\tilde{\boldsymbol{J}}_n{}^{n-m+1\to n}$ includes the last m column vectors chosen from $\tilde{\boldsymbol{J}}_n$ ($\tilde{\boldsymbol{J}}_n = \boldsymbol{J}_n$), which is defined as

$$\tilde{\boldsymbol{J}}_n{}^{n-m+1\to n} = [\tilde{\boldsymbol{j}}_{n,n-m+1}, \cdots, \tilde{\boldsymbol{j}}_{n,n}] \tag{6}$$

For easily understanding the intention of (5) in robotics field, Fig.1 shows a 4-link redundant manipulator with a given shape in plane ($m=m_p=2, m_o=0$). In Fig.1, the directions of four ratational axices are parallel ($^0\boldsymbol{z}_1//^0\boldsymbol{z}_2//^0\boldsymbol{z}_3//^0\boldsymbol{z}_4$) and $^0\boldsymbol{z}_i = {}^0\boldsymbol{R}_i\boldsymbol{e}_z$ where $^0\boldsymbol{R}_i$ is rotation matrix denoting the relation between Σ_0 and Σ_i, $\boldsymbol{e}_z=[0,0,1]^T$. $^0\boldsymbol{p}_{E,3}$ and $^0\boldsymbol{p}_{E,4}$ are described by broken lines ($^0\boldsymbol{p}_{E,k}$ denotes the position vector from the origin of Σ_k to the end-effector with respect to Σ_0). $^0\boldsymbol{p}_{2,1}$, $^0\boldsymbol{p}_{3,1}$ and $^0\boldsymbol{p}_{3,2}$ are described by dotted lines ($^0\boldsymbol{p}_{i+1,k}$ denotes the position vector from the origin of Σ_k to the one of Σ_i with respect to Σ_0). $\tilde{\boldsymbol{j}}_{1,1}$, $\tilde{\boldsymbol{j}}_{2,1}$, $\tilde{\boldsymbol{j}}_{2,2}$, $\tilde{\boldsymbol{j}}_{4,3}$ and $\tilde{\boldsymbol{j}}_{4,4}$ are described by solid lines. In addition, we define $sin(q_1)$ and $sin(q_1+q_2)$ by S_1 and S_{12}, $cos(q_1)$ and $cos(q_1+q_2)$ are C_1 and C_{12} and so on.

According to "*Assumptions*(a)", we can obtain

$$rank(\tilde{\boldsymbol{J}}_4{}^{3\to 4}) = rank([\tilde{\boldsymbol{j}}_{4,3}, \tilde{\boldsymbol{j}}_{4,4}]) = 2 \tag{7}$$

(7) indicates that $\tilde{\boldsymbol{j}}_{4,3}$ and $\tilde{\boldsymbol{j}}_{4,4}$ are independent.

According to "*Assumptions*(b)", we can obtain

$$\begin{cases} rank(\tilde{\boldsymbol{J}}_1) = rank([\tilde{\boldsymbol{j}}_{1,1}]) = 1 \\ rank(\tilde{\boldsymbol{J}}_2) = rank([\tilde{\boldsymbol{j}}_{2,1}, \tilde{\boldsymbol{j}}_{2,2}]) = 2 \end{cases} \tag{8}$$

(8) indicates that $\tilde{\boldsymbol{j}}_{1,1}$ is not zero vector and $\tilde{\boldsymbol{j}}_{2,1}$ and $\tilde{\boldsymbol{j}}_{2,1}$ are independent.

(7) and (8) are mathematical denotation. Now, we will explain the meaning of them in robotics field. Assuming $l_1=l_2=l_3=l_4=1[m]$,

$$\tilde{\boldsymbol{J}}_1 = \begin{bmatrix} -S_1 \\ C_1 \end{bmatrix} \tag{9}$$

Obviously, always $rank(\tilde{\boldsymbol{J}}_1)=1$ regardless of q_1.

$$\tilde{\boldsymbol{J}}_2 = \begin{bmatrix} -S_1-S_{12} & -S_{12} \\ C_1+C_{12} & C_{12} \end{bmatrix} \tag{10}$$

Obviously, $rank(\tilde{\boldsymbol{J}}_2)=2$ only if when $q_2 \neq 0$.

$$\tilde{\boldsymbol{J}}_4{}^{3\to 4} = \begin{bmatrix} -S_{234}-S_{1234} & -S_{1234} \\ C_{234}+C_{1234} & C_{1234} \end{bmatrix} \tag{11}$$

Obviously, $rank(\tilde{\boldsymbol{J}}_4{}^{3\to 4}) = 2$ only if when $q_4 \neq 0$. According to above discussion, in this example, it is called "Local Non-singular Configuration"when $q_2 \neq 0 \cap q_4 \neq 0$.

3·2 Results

By "*Assumptions*"(5), we can obtain "*Results*"as When $n \geq 2m$,

$$rank(^1\boldsymbol{M}_i) = \begin{cases} i \ (1 \leq i < m) \\ m \ (m \leq i \leq n-m) \\ n-i \sim m \ (n-m < i \leq n-2) \\ 1 \sim m-1 \ (i=n-1) \end{cases} \tag{12}$$

When $n < 2m$,

$$rank(^1\boldsymbol{M}_i) = \begin{cases} i \ (1 \leq i < n-m) \\ n-m \ (n-m \leq i \leq m) \\ n-i \sim n-m \ (m < i \leq n-1) \end{cases} \tag{13}$$

3·3 Proofs of (12) and (13)

We start these proofs by decomposing $^1\boldsymbol{M}_i$. Here, firstly we divide $\boldsymbol{V}_{n-m}$ in (4) as

$$\boldsymbol{V}_{n-m} = \begin{matrix} i \\ n-i \end{matrix} \overset{n-m}{\begin{pmatrix} \boldsymbol{V}_{i,(n-m)} \\ \boldsymbol{V}_{(n-i),(n-m)} \end{pmatrix}} \tag{14}$$

According to (2), (4) and (14), $^1\boldsymbol{M}_i$ can be decomposed by

$$\begin{aligned} ^1\boldsymbol{M}_i &= \boldsymbol{J}_i\boldsymbol{L}_n \\ &= m \overset{\begin{matrix} i & n-i \end{matrix}}{\begin{pmatrix} \tilde{\boldsymbol{J}}_i & \boldsymbol{0} \end{pmatrix}} \ n \overset{n-m}{\begin{pmatrix} \boldsymbol{V}_{n-m} \end{pmatrix}} \ n-m \overset{n}{\begin{pmatrix} \boldsymbol{V}^T_{n-m} \end{pmatrix}} \\ &= m \overset{i}{\begin{pmatrix} \tilde{\boldsymbol{J}}_i \end{pmatrix}} \ i \overset{n-m}{\begin{pmatrix} \boldsymbol{V}_{i,(n-m)} \end{pmatrix}} \ n-m \overset{n}{\begin{pmatrix} \boldsymbol{V}^T_{n-m} \end{pmatrix}} \end{aligned} \tag{15}$$

Then, we can obtain

$$\begin{aligned} rank(^1\boldsymbol{M}_i) &= rank(\tilde{\boldsymbol{J}}_i\boldsymbol{V}_{i,(n-m)}\boldsymbol{V}^T_{n-m}) \\ &\geq rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}\boldsymbol{V}^T_{n-m}) - i \\ &\geq rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}) \\ &\quad + rank(\boldsymbol{V}^T_{n-m}) - i - (n-m) \\ &= rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}) + (n-m) \\ &\quad -i-(n-m) \\ &= rank(\tilde{\boldsymbol{J}}_i) + rank(\boldsymbol{V}_{i,(n-m)}) - i \end{aligned} \tag{16}$$

and

$$\begin{aligned} rank({}^1\boldsymbol{M}_i) &= rank(\tilde{\boldsymbol{J}}_i\,\boldsymbol{V}_{i,(n-m)}\,\boldsymbol{V}_{n-m}^T) \\ &\le min\{rank(\tilde{\boldsymbol{J}}_i),\, rank(\boldsymbol{V}_{i,(n-m)}), \\ &\quad rank(\boldsymbol{V}_{n-m}^T)\} \\ &= min\{rank(\tilde{\boldsymbol{J}}_i),\, rank(\boldsymbol{V}_{i,(n-m)}), \\ &\quad n-m\} \end{aligned} \tag{17}$$

Then, inputting "*Assumption*(b)"and (55) into (16) and (17) (the proof of (55) is shown in "*Appendix*"), we can obtain

$$min\{i,m\} + min\{i,n-m\} - i \le rank({}^1\boldsymbol{M}_i) \le min\{i,m,n-m\} \tag{18}$$

When $n \ge 2m$, we will roughly discuss the four conditions ($1 \le i < m$, $m \le i \le n-m$, $n-m < i \le n-2$ and $i = n-1$) respectively as following.

(1): When $1 \le i < m$, by inputting this condition into (18), we can obtain

$$rank({}^1\boldsymbol{M}_i) = i \tag{19}$$

(2): When $m \le i \le n-m$, by inputting this condition into (18), we can obtain

$$rank({}^1\boldsymbol{M}_i) = m \tag{20}$$

(3): When $n-m < i \le n-2$, by inputting this condition into (18), we can obtain

$$n - i \le rank({}^1\boldsymbol{M}_i) \le m \tag{21}$$

(4): When $i = n-1$, we can obtain

$${}^1\boldsymbol{M}_{n-1} = \tilde{\boldsymbol{J}}_{n-1}\,\boldsymbol{V}_{(n-1),(n-m)}\,\boldsymbol{V}_{n-m}^T \tag{22}$$

By inputting "*Assumption*(b)"and (55) into (16), we can obtain

$$1 \le rank({}^1\boldsymbol{M}_{n-1}) \tag{23}$$

In addition, ${}^1\boldsymbol{M}_{n-1}$ can be rewritten as

$$\begin{aligned} {}^1\boldsymbol{M}_{n-1} &= \boldsymbol{J}_{n-1}\,\boldsymbol{L}_n \\ &= (\boldsymbol{J}_n - \Delta\boldsymbol{J}_n)\,\boldsymbol{L}_n \\ &= -\Delta\boldsymbol{J}_n\,\boldsymbol{L}_n \end{aligned} \tag{24}$$

In (24), we can prove $rank(\Delta\boldsymbol{J}_n) \le m-1$ (definition of $\Delta\boldsymbol{J}_n$ and proof are skipped). And because $rank(\boldsymbol{L}_n) = n-m \ge m-1$ from (4), so, we can obtain

$$rank({}^1\boldsymbol{M}_{n-1}) \le m-1 \tag{25}$$

Then, from (23) and (25), we can obtain

$$1 \le rank({}^1\boldsymbol{M}_{n-1}) \le m-1 \tag{26}$$

In this way, the result (12) is proved.

When $n < 2m$, we will roughly discuss the three conditions ($1 \le i < n-m$, $n-m \le i \le m$ and $m < i \le n-1$) respectively as following.

(1): When $1 \le i < n-m$, by inputting this condition into (18), we can obtain

$$rank({}^1\boldsymbol{M}_i) = i \tag{27}$$

(2): When $n-m \le i \le m$, by inputting this condition into (18), we can obtain

$$rank({}^1\boldsymbol{M}_i) = n-m \tag{28}$$

(3): When $m < i \le n-1$, by inputting this condition into (18), we can obtain

$$n - i \le rank({}^1\boldsymbol{M}_i) \le n-m \tag{29}$$

In this way, the result (13) is proved.

4. Conclusion

In this paper, we analyse the avoidance manipulability of redundant manipulators in all kinds of spaces ($m = 2, 3, \cdots, 6$). Moreover, we find the assumptions of manipulator's shape for ensuring the optimal shape-changing ability of manipulator as much as possible. We think that it is meaningful to assess the shape-changing ability to improve the structure at the first step of design for a new robot.

5. Appendix

5·1 Proof of $rank(\boldsymbol{V}_{m,m}) = m$ ($rank(\boldsymbol{B}) = m$)

If $\boldsymbol{V}$ is denoted by

$$\begin{aligned} \boldsymbol{V} &= \overset{}{n}\begin{pmatrix} \overset{m}{\boldsymbol{V}_m} & \overset{n-m}{\boldsymbol{V}_{n-m}} \end{pmatrix} \\ &= \begin{matrix} n-m \\ m \end{matrix}\begin{pmatrix} \overset{m}{\boldsymbol{V}_{(n-m),m}} & \overset{n-m}{\boldsymbol{V}_{(n-m),(n-m)}} \\ \boldsymbol{V}_{m,m} & \boldsymbol{V}_{m,(n-m)} \end{pmatrix} \\ &= \begin{matrix} n-m \\ m \end{matrix}\begin{pmatrix} \overset{m}{\boldsymbol{A}} & \overset{n-m}{\boldsymbol{C}} \\ \boldsymbol{B} & \boldsymbol{D} \end{pmatrix} \end{aligned} \tag{30}$$

According to "*Assumption*(b)", we can obtain $rank(\boldsymbol{J}_n) = m$, so, $\boldsymbol{J}_n$ can be decomposed by

$$\boldsymbol{J}_n = \boldsymbol{U}_m\boldsymbol{\Sigma}_m\boldsymbol{V}_m^T = \boldsymbol{R}_m\boldsymbol{V}_m^T \tag{31}$$

In (31), because $rank(\boldsymbol{U}_m) = m$ and $rank(\boldsymbol{\Sigma}_m) = m$, so $rank(\boldsymbol{R}_m) = rank(\boldsymbol{U}_m\boldsymbol{\Sigma}_m) = m$. Then, according to (31), we can obtain

$$\boldsymbol{V}_m^T = \boldsymbol{R}_m^{-1}\boldsymbol{J}_n \tag{32}$$

(32) can be rewritten by

$$[\boldsymbol{V}_{(n-m),m}^T,\ \boldsymbol{V}_{m,m}^T] = \boldsymbol{R}_m^{-1}\boldsymbol{J}_n \tag{33}$$

According to (33), we can obtain

$$\boldsymbol{V}_{m,m}^T = \boldsymbol{R}_m^{-1}\tilde{\boldsymbol{J}}_n^{n-m+1\to n} \tag{34}$$

In (34), because $rank(\boldsymbol{R}_m^{-1}) = m$ and "$Assumption(a)$"($rank(\tilde{\boldsymbol{J}}_n^{n-m+1\to n}) = m$), we can obtain

$$rank(\boldsymbol{V}_{m,m}^T) = m \tag{35}$$

Further, we can obtain

$$rank(\boldsymbol{V}_{m,m}) = m \tag{36}$$

5·2 Proof of $rank(\boldsymbol{V}_{(n-m),(n-m)}) = n - m$ ($rank(\boldsymbol{C}) = n - m$)

According to (30), we can obtain that

$$\boldsymbol{V}^T\boldsymbol{V} = \begin{bmatrix} \boldsymbol{A}^T\boldsymbol{A} + \boldsymbol{B}^T\boldsymbol{B} & \boldsymbol{A}^T\boldsymbol{C} + \boldsymbol{B}^T\boldsymbol{D} \\ \boldsymbol{C}^T\boldsymbol{A} + \boldsymbol{D}^T\boldsymbol{B} & \boldsymbol{C}^T\boldsymbol{C} + \boldsymbol{D}^T\boldsymbol{D} \end{bmatrix} \tag{37}$$

and

$$\boldsymbol{V}\boldsymbol{V}^T = \begin{bmatrix} \boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{C}\boldsymbol{C}^T & \boldsymbol{A}\boldsymbol{B}^T + \boldsymbol{C}\boldsymbol{D}^T \\ \boldsymbol{B}\boldsymbol{A}^T + \boldsymbol{D}\boldsymbol{C}^T & \boldsymbol{B}\boldsymbol{B}^T + \boldsymbol{D}\boldsymbol{D}^T \end{bmatrix} \tag{38}$$

And because of the condition that

$$\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}_n \tag{39}$$

So, from (37), we can obtain

$$\boldsymbol{A}^T\boldsymbol{A} + \boldsymbol{B}^T\boldsymbol{B} = \boldsymbol{I}_m \tag{40}$$

Because of the condition that

$$\boldsymbol{V}\boldsymbol{V}^T = \boldsymbol{I}_n \tag{41}$$

So, from (38), we can obtain

$$\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{C}\boldsymbol{C}^T = \boldsymbol{I}_{n-m} \tag{42}$$

$\boldsymbol{A}^T$ and $\boldsymbol{A}$ can be decomposed by

$$\boldsymbol{A}^T = {}^A\boldsymbol{U}{}^A\boldsymbol{\Sigma}{}^A\boldsymbol{V}^T \tag{43}$$

and

$$\boldsymbol{A} = {}^A\boldsymbol{V}{}^A\boldsymbol{\Sigma}^{T\,A}\boldsymbol{U}^T \tag{44}$$

In (43) and (44), ${}^A\boldsymbol{U}$ is $m\times m$ matrix satisfying ${}^A\boldsymbol{U}{}^A\boldsymbol{U}^T = {}^A\boldsymbol{U}^{T\,A}\boldsymbol{U} = \boldsymbol{I}_m$, ${}^A\boldsymbol{\Sigma}$ is $m\times(n-m)$ matrix including singular values of $\boldsymbol{A}$, ${}^A\boldsymbol{V}$ is $(n-m)\times(n-m)$ matrix satisfying ${}^A\boldsymbol{V}{}^A\boldsymbol{V}^T = {}^A\boldsymbol{V}^{T\,A}\boldsymbol{V} = \boldsymbol{I}_{n-m}$. Then, we can obtain

$$\boldsymbol{A}^T\boldsymbol{A} = {}^A\boldsymbol{U}{}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^{T\,A}\boldsymbol{U}^T \tag{45}$$

and

$$\boldsymbol{A}\boldsymbol{A}^T = {}^A\boldsymbol{V}{}^A\boldsymbol{\Sigma}^{T\,A}\boldsymbol{\Sigma}{}^A\boldsymbol{V}^T \tag{46}$$

According to (40) and (45), we can obtain

$$\begin{aligned}\boldsymbol{B}^T\boldsymbol{B} &= \boldsymbol{I}_m - \boldsymbol{A}^T\boldsymbol{A} \\ &= {}^A\boldsymbol{U}{}^A\boldsymbol{U}^T - {}^A\boldsymbol{U}{}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^{T\,A}\boldsymbol{U}^T \\ &= {}^A\boldsymbol{U}(\boldsymbol{I}_m - {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T){}^A\boldsymbol{U}^T\end{aligned} \tag{47}$$

Further, we can obtain

$$\boldsymbol{I}_m - {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T = {}^A\boldsymbol{U}^T\boldsymbol{B}^T\boldsymbol{B}{}^A\boldsymbol{U} \tag{48}$$

In (48), because $rank(\boldsymbol{B}) = m$ and $rank({}^A\boldsymbol{U}) = m$, so we can obtain

$$rank(\boldsymbol{I}_m - {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T) = m \tag{49}$$

Then, according to (42) and (46), we can obtain

$$\begin{aligned}\boldsymbol{C}\boldsymbol{C}^T &= \boldsymbol{I}_{n-m} - \boldsymbol{A}\boldsymbol{A}^T \\ &= {}^A\boldsymbol{V}{}^A\boldsymbol{V}^T - {}^A\boldsymbol{V} \begin{matrix} m \\ n-2m \end{matrix} \overset{\begin{matrix} m & \quad n-2m \end{matrix}}{\begin{pmatrix} {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T & \emptyset \\ \emptyset & \emptyset \end{pmatrix}} {}^A\boldsymbol{V}^T \\ &= {}^A\boldsymbol{V}(\boldsymbol{I}_{n-m} - \begin{matrix} m \\ n-2m \end{matrix} \overset{\begin{matrix} m & \quad n-2m \end{matrix}}{\begin{pmatrix} {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T & \emptyset \\ \emptyset & \emptyset \end{pmatrix}}){}^A\boldsymbol{V}^T \\ &= {}^A\boldsymbol{V} \begin{matrix} m \\ n-2m \end{matrix} \overset{\begin{matrix} m & \quad n-2m \end{matrix}}{\begin{pmatrix} \boldsymbol{I}_m - {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T & \emptyset \\ \emptyset & \boldsymbol{I}_{n-2m} \end{pmatrix}} {}^A\boldsymbol{V}^T\end{aligned} \tag{50}$$

In (50), because of (49), we can obtain

$$rank(\begin{bmatrix} \boldsymbol{I}_m - {}^A\boldsymbol{\Sigma}{}^A\boldsymbol{\Sigma}^T & \emptyset \\ \emptyset & \boldsymbol{I}_{n-2m} \end{bmatrix}) = n - m \tag{51}$$

and because $rank({}^A\boldsymbol{V}) = n - m$ and (51), we can obtain $rank(\boldsymbol{C}\boldsymbol{C}^T) = n - m$. Further, $rank(\boldsymbol{C}) = n - m$, that is

$$rank(\boldsymbol{V}_{(n-m),(n-m)}) = n - m \tag{52}$$

5·3 $rank(\boldsymbol{V}_{i,(n-m)})$

When $1 \leq i < n - m$, we can obtain the relation between $\boldsymbol{V}_{i,(n-m)}$ and $\boldsymbol{V}_{(n-m),(n-m)}$ as

$$\boldsymbol{V}_{(n-m),(n-m)} = \begin{bmatrix} \boldsymbol{V}_{i,(n-m)} \\ \boldsymbol{V}_{(n-m-i),(n-m)} \end{bmatrix} \tag{53}$$

According to (52) and (53), $\boldsymbol{V}_{(n-m),(n-m)}$ is $(n-m)\times(n-m)$ matrix and $rank(\boldsymbol{V}_{(n-m),(n-m)}) = n - m$, $\boldsymbol{V}_{i,(n-m)}$ is $i\times(n-m)$ matrix and $\boldsymbol{V}_{i,(n-m)}$ is one part of $\boldsymbol{V}_{(n-m),(n-m)}$. So, it is obvious that the i row vectors of $\boldsymbol{V}_{i,(n-m)}$ are independent and we can obtain $rank(\boldsymbol{V}_{i,(n-m)}) = i$.

When $n - m \leq i \leq n$, we can obtain the relation between $\boldsymbol{V}_{i,(n-m)}$ and $\boldsymbol{V}_{(n-m),(n-m)}$ as

$$\boldsymbol{V}_{i,(n-m)} = \begin{bmatrix} \boldsymbol{V}_{(n-m),(n-m)} \\ \boldsymbol{V}_{(i-n+m),(n-m)} \end{bmatrix} \tag{54}$$

According to (52) and (54), $\boldsymbol{V}_{(n-m),(n-m)}$ is one part of $\boldsymbol{V}_{i,(n-m)}$. So, we can obtain $rank(\boldsymbol{V}_{i,(n-m)}) = n - m$.

In this way, we can summarize the very important conclusion about $rank(\boldsymbol{V}_{i,(n-m)})$ $(i = 1, 2, \cdots, n)$ as follows:

$$rank(\boldsymbol{V}_{i,(n-m)}) = min\{i, n - m\} \tag{55}$$

[1] Tsuneo Yoshikawa, "Manipulability of Robot Mechanisms," The International Journal of Robotics Research, 4, 2, pp.3-9, 1985.